Math 2040 C Week 10

Let V be an inner product space, $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$

Orthogonal Complement

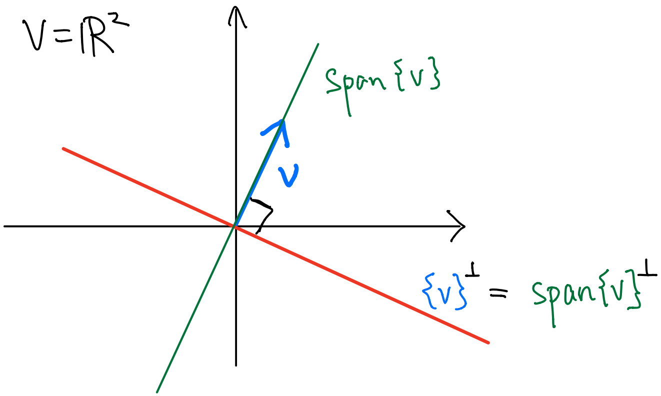
Defn 6.45 Let $S \subseteq V$ be a subset.

Define the orthogonal complement of S to be

$$S^{\perp} = \{ v \in V : \langle v, u \rangle = 0 \ \forall u \in S \}$$

Rmk $\langle v, u \rangle = 0 \Leftrightarrow \langle u, v \rangle = 0$

eg $V = \mathbb{R}^2$



Prop 6.46 Let $S \subseteq V$ be a subset

1. $S^{\perp}$ is a subspace of V
2. $\{0\}^{\perp} = V$, $V^{\perp} = \{0\}$
3. $S \cap S^{\perp} \subseteq \{0\}$
4. $S_1 \subseteq S_2 \subseteq V \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$
5. $(\operatorname{span} S)^{\perp} = S^{\perp}$

Pf of 1, 3, 5

1. Since $\langle \vec{0}, u \rangle = 0 \ \forall u \in S$, $\vec{0} \in S^{\perp}$

   Suppose $v, w \in S^{\perp}$, $\lambda \in \mathbb{F}$. Then $\forall u \in S$,

   $\langle v+w, u \rangle = \langle v, u \rangle + \langle w, u \rangle = 0 + 0 = 0$

   $\langle \lambda v, u \rangle = \lambda \langle v, u \rangle = 0$

$\therefore S^{\perp}$ is closed under addition, scalar multiplication

$\therefore S^{\perp}$ is a subspace.

3. If $v \in S \cap S^{\perp}$, then $\langle v, v \rangle = 0 \Rightarrow v = \vec{0}$

⑤ $S \subseteq \operatorname{span} S \Rightarrow (\operatorname{span} S)^\perp \subseteq S^\perp$ by ④

Suppose $v \in S^\perp$. Let $u \in \operatorname{span} S$. Then

$u = a_1 u_1 + \cdots + a_k u_k$ for some $a_i \in \mathbb{F}$, $u_i \in S$

$\Rightarrow \langle v, u \rangle = \langle v, \sum_{i=1}^{k} a_i u_i \rangle = \sum_{i=1}^{k} \overline{a_i} \langle v, u_i \rangle = 0$

$\Rightarrow v \in \operatorname{span} S^\perp$

$\boxed{\text{Prop 6.47 Let } U \subseteq V \text{ be subspace, } \dim U < \infty. \text{ Then } V = U \oplus U^\perp}$

Pf We first show $V = U + U^\perp$

Let $\alpha = \{e_1, \cdots, e_m\}$ be an orthonormal basis of U

For any $v \in V$, let

$$\boxed{u = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m}$$

and $w = v - u$

Then for any $j = 1, \cdots, m$,

$$\begin{aligned}
\langle w, e_j \rangle &= \langle v - u, e_j \rangle \\
&= \langle v, e_j \rangle - \langle u, e_j \rangle \\
&= \langle v, e_j \rangle - \Big\langle \sum_{i=1}^{m} \langle v, e_i \rangle e_i, e_j \Big\rangle \\
&= \langle v, e_j \rangle - \sum_{i=1}^{m} \langle v, e_i \rangle \langle e_i, e_j \rangle \\
&= \langle v, e_j \rangle - \langle v, e_j \rangle = 0
\end{aligned}$$

$\therefore \quad w \in \alpha^\perp = (\operatorname{span} \alpha)^\perp = U^\perp$

Hence, $v = u + w$ with

$u \in \operatorname{span} \alpha = U$ and $w \in U^\perp$

$\therefore \quad V = U + U^\perp$

Also, by Prop 6.46, $U \cap U^\perp \subseteq \{0\}$

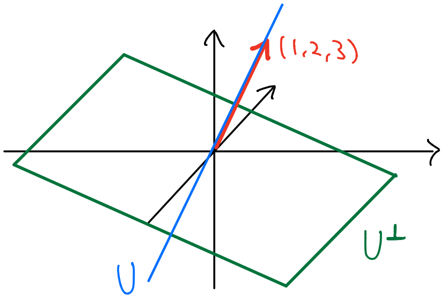
$U, U^\perp$ are subspace $\Rightarrow \{0\} \subseteq U \cap U^\perp$

$\therefore U \cap U^\perp = \{0\}$ and $V = U \oplus U^\perp$.

Cor 6.50 Suppose $\dim V < \infty$, U is a subspace of V. Then $\dim U^\perp = \dim V - \dim U$

eg $V = \mathbb{R}^3$, $U = \operatorname{span}\{(1,2,3)\}$

$$U^\perp = \{(x,y,z) \in \mathbb{R}^3 : x + 2y + 3z = 0\}$$



Prop 6.51 Suppose $U \subseteq V$, $\dim U < \infty$. Then

$$(U^\perp)^\perp = U$$

Pf ① Show $U \subseteq (U^\perp)^\perp$: Suppose $v \in U$. Then

$\forall w \in U^\perp$, $\langle v, w\rangle = \overline{\langle w, v\rangle} = 0$ since $w \in U^\perp$, $v \in U$.

$\Rightarrow v \in (U^\perp)^\perp$ and so $U \subseteq (U^\perp)^\perp$

② Show $(U^\perp)^\perp \subseteq U$. Suppose $v \in (U^\perp)^\perp$. Then by $V = U \oplus U^\perp$, $\exists$ unique $u \in U$, $w \in U^\perp$ such that $v = u + w$.

$v \in (U^\perp)^\perp$, $w \in U^\perp$, $u \in U \subseteq (U^\perp)^\perp$

$$\Rightarrow 0 = \langle v, w\rangle = \langle u+w, w\rangle = \langle u, w\rangle + \langle w, w\rangle$$
$$= \langle w, w\rangle$$

$\Rightarrow w = \vec{0}$, $v = u \in U$ and so $(U^\perp)^\perp \subseteq U$

Hence, $U = (U^\perp)^\perp$.

Rmk $\dim U < \infty$ is needed in 6.47 and 6.51:

Consider $V = \{x \in \mathbb{R}^\infty : \sum x_i^2 < \infty\}$ with

$$\langle x, y\rangle = \sum_{i=1}^\infty x_i y_i$$

Let $U = \operatorname{span}\{e_i : i \in \mathbb{N}\}$. Then $U^\perp = \{\vec{0}\}$

Hence ① $U \oplus U^\perp = U \neq V$

② $(U^\perp)^\perp = \{0\}^\perp = V \neq U$

Orthogonal Projection

If $U \subseteq W$, $\dim U < \infty$, then $V = U \oplus U^\perp$.

Hence, we can define:

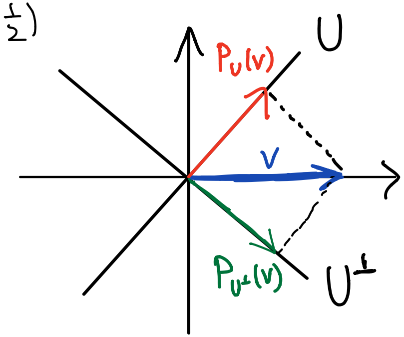
Defn 6.53 Suppose $U \subseteq V$, $\dim U < \infty$. Define the orthogonal projection of V onto U to be the map $P_U : V \to V$ as follows:

For any $v \in V$, $\exists$ unique $u \in U$ and $w \in U^\perp$ s.t. $v = u + w$. Define $P_U(v) = u$.

eg $V = \mathbb{R}^2$, $U = \text{span}\{(1,1)\}$, $U^\perp = \text{span}\{(1,-1)\}$.

$$\underset{\substack{\shortparallel \\ v}}{(1,0)} = \underset{\substack{\cap \\ U}}{\left(\tfrac{1}{2}, \tfrac{1}{2}\right)} + \underset{\substack{\cap \\ U^\perp}}{\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)}$$

$\therefore P_U(1,0) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$



Prop 6.55 Let $U \subseteq V$, $\dim U < \infty$

a. $P_U \in \mathcal{L}(V)$

b. $P_U(u) = u \quad \forall u \in U$

c. $P_U(w) = \vec{0} \quad \forall w \in U^\perp$

d. range $P_U = U$

e. null $P_U = U^\perp$

f. $v - P_U(v) \in U^\perp$

g. $P_U^2 = P_U$

h. $\|P_U(v)\| \leq \|v\|$

Pf of a Suppose $v_1, v_2 \in V$ and

$v_1 = u_1 + w_1$, $v_2 = u_2 + w_2$, $u_i \in U$ $w_i \in U^\perp$

Then $v_1 + v_2 = (u_1 + u_2) + (w_1 + w_2)$

with $u_1 + u_2 \in U$, $w_1 + w_2 \in U^\perp$

$\therefore P_U(v_1 + v_2) = u_1 + u_2 = P_U(v_1) + P_U(v_2)$

Similarly $P_U(\lambda v) = \lambda P_U(v) \quad \forall v \in V, \lambda \in \mathbb{F}$.

Hence, P_U is linear

Pf of h Let $v = u + w$, $u \in U$ and $w \in U^\perp$

Then $\|v\|^2 = \|u\|^2 + \|w\|^2 \geq \|P_U(v)\|^2 \Rightarrow$ h

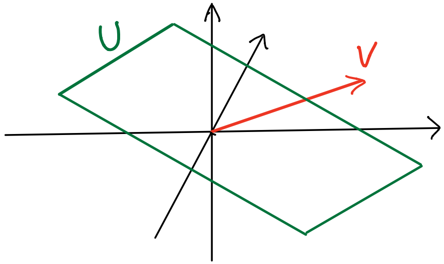
Prop 6.55 i Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of U. Then

$$P_U(v) = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$$

Pf It follows from the construction of u in pf of Prop 6.47 $(V = U \oplus U^{\perp})$

Minimizing Property of Orthogonal Projection

Given $U \subseteq V$ and $v \in V$. Want to find $u \in U$ s.t. $\|v-u\|$ is minimized.



Q Which $u \in U$ is closest to v?

Ans $u = P_U(v)$!

Prop 6.57 Let $U \subseteq V$, $\dim U < \infty$, $v \in V$, $u \in U$

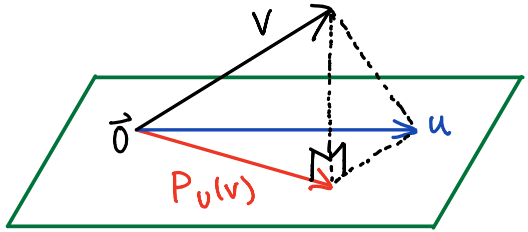
Then $\|v - P_U(v)\| \leq \|v - u\|$

with equality holds $\Leftrightarrow$ $u = P_U(v)$

Pf For any $u \in U$,

$$v - u = (v - P_U(v)) + (P_U(v) - u)$$

Note $v - P_U(v) \in U^{\perp}$ and $P_U(v) - u \in U$



$$\therefore \|v-u\|^2 = \|v - P_U(v)\|^2 + \|P_U(v) - u\|^2$$
$$\geq \|v - P_U(v)\|^2$$

$$\therefore \|v-u\| \geq \|v - P_U(v)\|$$

Equality $\Leftrightarrow$ $\|P_U(v) - u\| = 0$ $\Leftrightarrow$ $u = P_U(v)$

eg Find the "best" polynomial of deg ≤ 2 to approximate $f(x) = |x|$ over $[-1,1]$

Let $V = C([-1,1])$

$= \{ f : [-1,1] \to \mathbb{R} : f \text{ is continuous} \}$

with $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$

Let $U = P_2(\mathbb{R}) \subseteq V$

Note $\dim U = 3$, $\dim V = \infty$

$\{1, x, x^2\}$ basis of U

$\downarrow$ Gram-Schmidt

$\{p_0, p_1, p_2\}$ orthonormal basis of U

with $p_0 = \sqrt{\frac{1}{2}}$ $\quad p_1 = \sqrt{\frac{3}{2}}\, x$ $\quad p_2 = \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right)$

Let $v = |x| \in V$. To find $P_U(v)$:

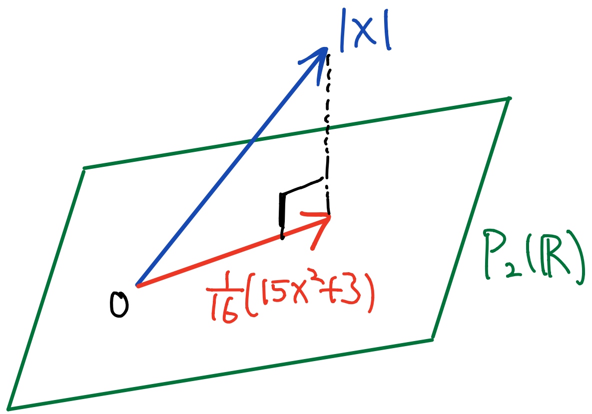
$\langle v, p_0 \rangle = \int_{-1}^{1} |x| \left(\frac{1}{\sqrt{2}} \right) dx = \sqrt{\frac{1}{2}}$

$\langle v, p_1 \rangle = \int_{-1}^{1} |x| \left(\sqrt{\frac{3}{2}}\, x \right) dx = 0$

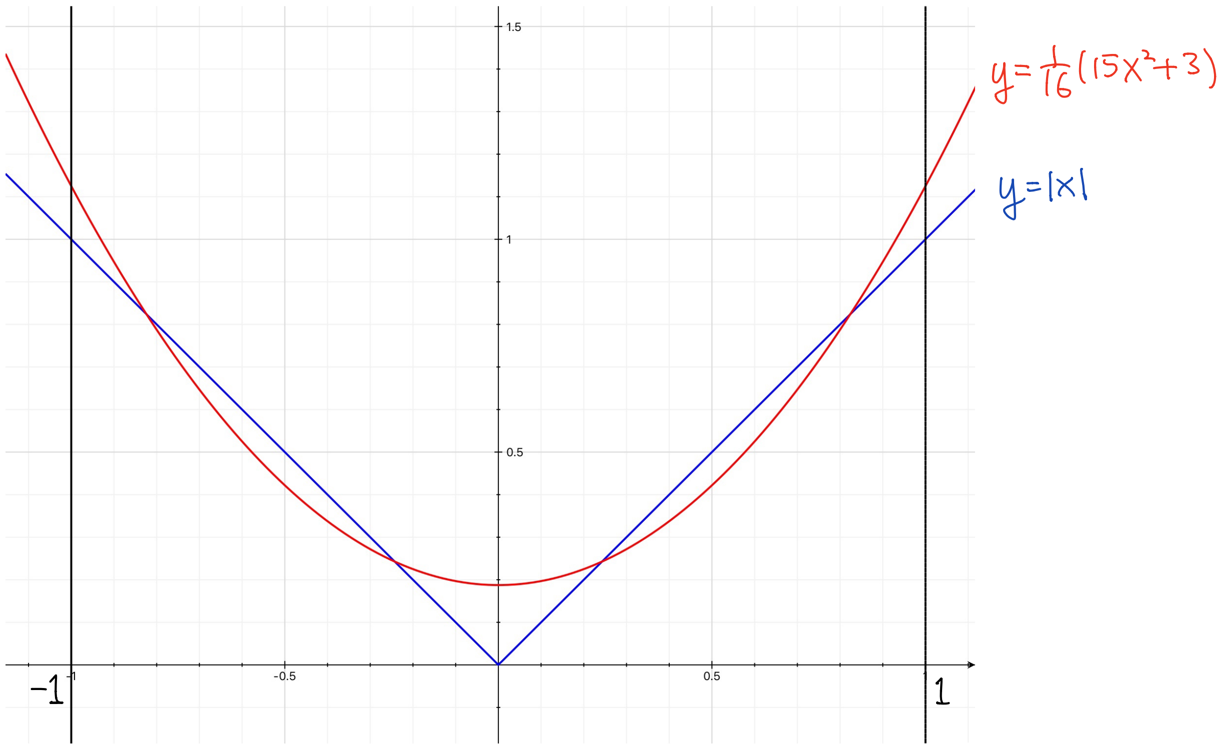
$\langle v, p_2 \rangle = \int_{-1}^{1} |x| \left[\sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right) \right] dx = \frac{1}{6} \sqrt{\frac{45}{8}}$

$\therefore P_U(v) = \sqrt{\frac{1}{2}}\, p_0 + 0\, p_1 + \frac{1}{6} \sqrt{\frac{45}{8}}\, p_2$

$= \frac{1}{16} (15x^2 + 3)$



<u>Rmk</u> From last example, we found that $\frac{1}{16}(15x^2+3)$ is the "best" polynomial of $\deg \leq 2$ to approximate $|x|$ on $[-1,1]$ in the sense that it minimize $\int_{-1}^{1}(|x|-p(x))^2\,dx$ among all polynomials $p(x)$ of $\deg \leq 2$



Linear Maps of Inner Product Spaces

Let V, W be inner product spaces.

Defn Suppose $T \in L(V,W)$.
The adjoint of T is a function $T^*: W \to V$ such that for any $v \in V$, $w \in W$

$$\underbrace{\langle T(v), w \rangle}_{\text{inner product in } W} = \underbrace{\langle v, T^*(w) \rangle}_{\text{inner product in } V}$$

Analog

Adjoint in $L(V,W)$ $\longleftrightarrow$ Conjugate in $\mathbb{C}$ $\longleftrightarrow$ Conjugate Transpose in $M_{m\times n}(\mathbb{C})$

Q T^* exists? unique?

A Yes if $\dim < \infty$.

Recall: Riesz Representation theorem

Thm $\dim V < \infty$, $\varphi \in L(V, \mathbb{F})$. Then $\exists$ unique $u \in V$ s.t. $\varphi(v) = \langle v, u \rangle$ $\forall v \in V$

Prop Let $\dim V < \infty$, $T \in L(V,W)$. Then $T^*: W \to V$ exists and is unique, linear

Pf. ① Existence

Let $w \in W$. We want to define $T^*(w)$.

Consider $g_w : V \to \mathbb{F}$ be defined by

$$g_w(v) = \langle T(v), w \rangle$$

Note g_w is linear. By Riesz Representation Thm, $\exists$ unique $w' \in V$ s.t. $g_w(v) = \langle v, w' \rangle$ $\forall v \in V$

Define $T^*(w) = w'$. Then

$$\langle T(v), w \rangle = g_w(v) = \langle v, w' \rangle = \langle v, T^*(w) \rangle$$

② Uniqueness

Suppose $S: W \to V$ is a function such that

$$\langle T(v), w \rangle = \langle v, S(w) \rangle \quad \forall v \in V, w \in W$$

Then $\langle v, T^*(w) \rangle = \langle T(v), w \rangle$

$$= \langle v, S(w) \rangle \quad \forall v, w$$

$\Rightarrow \langle v, T^*(w) - S(w) \rangle = 0 \quad \forall v, w$

Put $v = T^*(w) - S(w)$. Then

$$\langle T^*(w) - S(w), T^*(w) - S(w) \rangle = 0 \quad \forall w$$

$\Rightarrow T^*(w) - S(w) = \vec{0} \quad \forall w \in W$

$\Rightarrow T^* = S$ and T^* is unique.

Rmk Argument above shows

If $u, w \in V$ s.t. $\langle v, u \rangle = \langle v, w \rangle \ \forall v \in V$

then $u = w$

③ Linearity

Suppose $w_1, w_2 \in W$, $v \in V$

$$\begin{aligned} \langle v, T^*(w_1 + w_2) \rangle &= \langle T(v), w_1 + w_2 \rangle \\ &= \langle T(v), w_1 \rangle + \langle T(v), w_2 \rangle \\ &= \langle v, T^*(w_1) \rangle + \langle v, T^*(w_2) \rangle \\ &= \langle v, T^*(w_1) + T^*(w_2) \rangle \end{aligned}$$

v is arbitrary $\Rightarrow T^*(w_1 + w_2) = T^*(w_1) + T^*(w_2)$

Similarly, if $w \in W$, $\lambda \in \mathbb{F}$, $v \in V$

$$\begin{aligned} \langle v, T^*(\lambda w) \rangle &= \langle T(v), \lambda w \rangle \\ &= \bar{\lambda} \langle T(v), w \rangle \\ &= \bar{\lambda} \langle v, T^*(w) \rangle \\ &= \langle v, \lambda T^*(w) \rangle \end{aligned}$$

v is arbitrary $\Rightarrow T^*(\lambda w) = \lambda T^*(w)$

Notation For a complex matrix A, let

$$A^* = \overline{A}^T = \text{conjugate transpose of } A.$$

e.g.
$$\begin{bmatrix} 1+i & 2 & i \\ 0 & -i & 2+3i \end{bmatrix}^* = \begin{bmatrix} 1-i & 0 \\ 2 & i \\ -i & 2-3i \end{bmatrix}$$

Prop 7.10 Let $T \in L(V,W)$ and α, β be orthonormal basis of V, W respectively. Then

$$M(T^*, \beta, \alpha) = M(T, \alpha, \beta)^*$$

Pf Let $\alpha = \{e_1, \cdots, e_n\}$, $\beta = \{f_1, \cdots, f_m\}$

$$A = M(T, \alpha, \beta), \quad B = M(T^*, \beta, \alpha)$$

Note j-th column of $B = M(T^*(f_j), \alpha)$

By Prop 6.30,

$$T^*(f_j) = \langle T^*(f_j), e_1 \rangle e_1 + \cdots + \langle T^*(f_j), e_n \rangle e_n$$

$$\therefore B_{ij} = \langle T^*(f_j), e_i \rangle$$

i-th row
j-th column

$$= \overline{\langle e_i, T^*(f_j) \rangle}$$
$$= \overline{\langle T(e_i), f_j \rangle}$$
$$= \overline{A_{ji}}$$

$$\Rightarrow B = A^*$$

eg $T : \mathbb{C}^3 \to \mathbb{C}^2$,

$$T(z_1, z_2, z_3) = ((1+i) z_3, z_1 - i z_2)$$

Let α, β be standard basis of $\mathbb{C}^3, \mathbb{C}^2$ resp.

Then
$$M(T, \alpha, \beta) = \begin{bmatrix} 0 & 0 & 1+i \\ 1 & -i & 0 \end{bmatrix}$$

$$\therefore M(T^*, \beta) = M(T, \beta)^* = \begin{bmatrix} 0 & 1 \\ 0 & i \\ 1-i & 0 \end{bmatrix}$$

$$\Rightarrow T^*(z_1, z_2) = (z_2, i z_2, (1-i) z_1)$$

eg Consider $P_2(\mathbb{R}) \subseteq C([-1,1])$ and

$$T \in \mathcal{L}(P_2(\mathbb{R})), \quad Tp = p'$$

Let $\alpha = \{1, x, x^2\}$

Then $M(T,\alpha) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\cancel{\Rightarrow M(T^*,\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}}$$

$$\cancel{\therefore T^*(a+bx+cx^2) = ax + 2bx^2}$$

α is not orthonormal !

Ex Find the correct T^*.

Properties of adjoint

inner product space / $\mathbb{F}$

Prop 7.6 Let $T, T_1, T_2 \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(W,U)$.

Suppose all their adjoints exist. (True if $\dim < \infty$). Then

① $(T_1 + T_2)^* = T_1^* + T_2^*$

② $(\lambda T)^* = \bar{\lambda} T^*$

③ $(T^*)^* = T$

④ $I_V^* = I_V$

⑤ $(ST)^* = T^* S^*$

Pf of ① Let $w, v \in V$, then

$$\begin{aligned} \langle w, (T_1+T_2)^*(v)\rangle &= \langle (T_1+T_2)(w), v\rangle \\ &= \langle T_1(w) + T_2(w), v\rangle \\ &= \langle T_1(w), v\rangle + \langle T_2(w), v\rangle \\ &= \langle w, T_1^*(v)\rangle + \langle w, T_2^*(v)\rangle \\ &= \langle w, T_1^*(v) + T_2^*(v)\rangle \\ &= \langle w, (T_1^* + T_2^*)(v)\rangle \end{aligned}$$

w is arbitrary

so $(T_1+T_2)^*(v) = (T_1^* + T_2^*)(v)$ $\forall v \in V \Rightarrow$ ①

Prop 7.7 Let $\dim V, \dim W < \infty$, $T \in L(V,W)$.

Then ① $\operatorname{null} T^* = (\operatorname{range} T)^\perp$

② $\operatorname{range} T^* = (\operatorname{null} T)^\perp$

③ $\operatorname{null} T = (\operatorname{range} T^*)^\perp$

④ $\operatorname{range} T = (\operatorname{null} T^*)^\perp$

Pf For ①, let $w \in W$. Note

$$
\begin{aligned}
w \in \operatorname{null} T^* &\iff T^*(w) = 0_V \\
&\iff \langle v, T^*(w) \rangle = 0 \quad \forall v \in V \\
&\iff \langle T(v), w \rangle = 0 \quad \forall v \in V \\
&\iff w \in (\operatorname{range} T)^\perp
\end{aligned}
$$

For ④, $(\operatorname{null} T^*)^\perp = \left[(\operatorname{range} T)^\perp\right]^\perp \underset{\because \dim W < \infty}{=} \operatorname{range} T$

By replacing T with T^*, since $(T^*)^* = T$

we have ① $\Rightarrow$ ③

② $\Rightarrow$ ④